

GRAY CURVATURE IDENTITIES FOR ALMOST CONTACT METRIC MANIFOLDS

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ABSTRACT. The aim of this research is the study of Gray curvature identities, introduced by Alfred Gray in [7] for the class of almost hermitian manifolds. As known till now, there is no equivalent for the class of almost contact manifolds. For this purpose we use the Boothby-Wang fibration and the warped manifolds construction in order to establish which identities could be satisfied by an almost contact manifold. An almost hermitian manifold which satisfies one of the three Gray identities has rich topological and geometric properties.

Keywords and Phrases: almost Hermitian manifolds, almost contact metric manifolds, curvature identities, Boothby Wang fibration, cone metric, cosymplectic manifolds, Sasakian manifolds, generalized Heisenberg group.

Mathematics Subject Classification (2000): 53C15, 53C25, 53C55, 53B35, 53D15.

1. INTRODUCTION

In their paper [4], the authors defined $K_{i\varphi}$ -curvature identities ($i = 1, 2, 3$) for an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ by using the usual Hermitian structure on $M \times \mathbf{R}$ (the product manifold). It is known that both cosymplectic and Sasakian manifolds are natural odd-dimension versions for Kaehlerian manifolds. Gray proved in [7] that Kaehlerian manifolds satisfy K_i , $i = 1, 2, 3$ (curvature identities for almost Hermitian manifolds). In the same spirit, in [4] it is shown that cosymplectic manifolds satisfy $K_{i\varphi}$ -identities. We asked what happens with Sasakian manifolds? Recall that a Riemannian manifold (M, g) is Sasakian if the holonomy group of the metric cone on M : $(C(M) = \mathbf{R}_+ \times M, \tilde{g} = dt^2 + t^2g)$ reduces to a subgroup of $U(\frac{m+1}{2})$, i.e. $(C(M), \tilde{g})$ is Kaehlerian. (Here $m = \dim M$.) Inspired from this definition and from [4] we will give another approach of Gray curvature identities for almost contact metric manifolds.

1.1. Gray curvature identities. An almost Hermitian manifold (M, J, g) is said to satisfy the Gray curvature identities $(K1)$, $(K2)$ and respectively $(K3)$, if his Riemann Christoffel curvature tensor verifies

$$(K1) \quad R(X, Y, Z, W) = R(X, Y, JZ, JW)$$

$$(K2) \quad R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$$

$$(K3) \quad R(X, Y, Z, W) = R(JX, JY, JZ, JW)$$

for all vector fields X, Y, Z, W on $\chi(M)$. Throughout of this paper, the curvature tensor

Date: February 1, 2008.

This work was supported by Grant CEEX ET n. 5883/2006-2008 ANCS Romania.

is defined by $R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$, for all $X, Y, Z \in \chi(M)$ while the Riemann Christoffel curvature tensor is given by $R(X, Y, Z, W) = -g(R_{XY}Z, W)$.

2. WARPED PRODUCT MANIFOLDS

Singly warped products or simply *warped products* were first defined by Bishop & O'Neill in [1] in order to construct Riemannian manifolds with negative sectional curvature. Let (B, g_B) and (F, g_F) be Riemannian manifolds and let $b : B \rightarrow (0, \infty)$ be a smooth function. The warped product $\widetilde{M} = B \times_b F$ is the product manifold $B \times F$ endowed with the metric $\widetilde{g} = g_B \oplus b^2 g_F$. More precisely, if $\pi : B \times F \rightarrow B$ and $\tau : B \times F \rightarrow F$ are natural projections, the metric g is defined by

$$(1) \quad \widetilde{g} = \pi^* g_B + (b \circ \pi)^2 \tau^* g_F.$$

The function b is called *warping function*. If $b \equiv 1$, then we have a product manifold.

If X, Y are tangent to B and Z, W tangent to F , then the Levi Civita connection $\widetilde{\nabla}$ of \widetilde{M} is given by

$$(2) \quad \begin{cases} \widetilde{\nabla}_X Y = \nabla_X^B Y, & \widetilde{\nabla}_X Z = X(\ln b)Z \\ \widetilde{\nabla}_Z W = \nabla_Z^F W - b^2 g_F(Z, W) \nabla^B(\ln b) \end{cases}$$

where ∇^B and ∇^F are the Levi Civita connections on B , respectively on F , and $\nabla^B(\ln b)$ is the gradient of $\ln b$ with respect to the metric g_B .

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. Consider the warped product manifold $\widetilde{M} = \mathbf{R}_+ \times_t M$, where t is the global coordinate of \mathbf{R}_+ , i.e. the metric \widetilde{g} of \widetilde{M} is defined by

$$(3) \quad \widetilde{g} = dt^2 + t^2 g.$$

Define an endomorphism on $\chi(\widetilde{M})$ by

$$(4) \quad J\partial_t = -\frac{1}{t} \xi \quad JX = \varphi X + t\eta(X)\partial_t, \quad \forall X \in \chi(M)$$

where $\partial_t = \frac{d}{dt}$. For $\widetilde{X} = (a, X) \in \chi(\widetilde{M})$, $a \in C^\infty(\mathbf{R}_+)$, $X \in \chi(M)$ we have

$$(5) \quad J\widetilde{X} = J(a, X) = (t\eta(X), \varphi X - \frac{a}{t} \xi)$$

The proofs of the following propositions are straightforward.

Proposition 2.1. *J is an almost complex structure compatible with the metric \widetilde{g} .*

Proposition 2.2. *The Levi-Civita connection $\widetilde{\nabla}$ of \widetilde{g} is given by:*

$$(6) \quad \begin{cases} \widetilde{\nabla}_{\partial_t} \partial_t = 0, & \widetilde{\nabla}_X \partial_t = \widetilde{\nabla}_{\partial_t} X = \frac{1}{t} X \\ \widetilde{\nabla}_X Y = \nabla_X Y - t g(X, Y) \partial_t, & X, Y \in \chi(M) \end{cases}$$

Proposition 2.3. *The covariant derivative of J is given by:*

$$(7) \quad \begin{cases} (\widetilde{\nabla}_{\partial_t} J) \partial_t = (0, 0), & (\widetilde{\nabla}_{\partial_t} J) X = (0, 0) \\ (\widetilde{\nabla}_X J) \partial_t = (0, -\frac{1}{t} (\nabla_X \xi + \varphi X)) \\ (\widetilde{\nabla}_X J) Y = (t((\nabla_X \eta)(Y) - g(X, \varphi Y)), (\nabla_X \varphi)Y - g(X, Y)\xi + \eta(Y)X) \end{cases}$$

Corollary 2.4. *J is parallel if and only if*

$$(8) \quad \begin{cases} (\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, & (\nabla_X \eta)(Y) = g(X, \varphi Y) \\ \nabla_X \xi = -\varphi X, & X, Y \in \chi(M) \end{cases}$$

i.e. $(\tilde{M}, J, \tilde{g})$ is Kaehler if and only if $(M, \varphi, \xi, \eta, g)$ is Sasakian.

Proposition 2.5. *For the curvature of the manifold \tilde{M} we have*

$$(9) \quad \begin{cases} \tilde{R}(\partial_t, X)\partial_t = 0, \quad \tilde{R}(X, Y)\partial_t = 0, \quad \tilde{R}(\partial_t, X)Y = 0 \\ \tilde{R}(X, Y)Z = R(X, Y)Z - g(Y, Z)X + g(X, Z)Y \end{cases}$$

where \tilde{R} (respectively R) are the curvature tensors for \tilde{g} (respectively for g).

Proposition 2.6. *Moreover, the following relations hold:*

$$(10) \quad \begin{cases} \tilde{R}(\partial_t, X)(J\partial_t) = 0, \quad \tilde{R}(\partial_t, X)(JY) = 0 \\ \tilde{R}(X, Y)(J\partial_t) = -\frac{1}{t}[R(X, Y)\xi - \eta(Y)X + \eta(X)Y] \\ \tilde{R}(X, Y)(JZ) = R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y \end{cases}$$

In the following we compute expressions of the form $\tilde{g}(\tilde{R}(A, B)(JC), JD)$. The useful expressions are obtained in the following cases:

1. $\tilde{g}(\tilde{R}(X, Y)(J\partial_t), JW) = -t[g(R(X, Y)\xi, \varphi W) - \eta(Y)g(X, \varphi W) + \eta(X)g(Y, \varphi W)]$
2. $\tilde{g}(\tilde{R}(X, Y)(JZ), J\partial_t) = -t[\eta(R(X, Y)(\varphi Z)) - \eta(X)g(Y, \varphi Z) + \eta(Y)g(X, \varphi Z)]$
3. $\tilde{g}(\tilde{R}(X, Y)(JZ), JW) = t^2[g(R(X, Y)\varphi Z, \varphi W) - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W)].$

Theorem 2.7. *\tilde{M} is (K1) if and only if*

$$(11) \quad \begin{aligned} R(X, Y, Z, W) &= R(X, Y, \varphi Z, \varphi W) - g(X, \varphi Z)g(Y, \varphi W) + \\ &+ g(Y, \varphi Z)g(X, \varphi W) - g(Y, Z)g(X, W) + g(X, Z)g(Y, W). \end{aligned}$$

PROOF. \tilde{M} is (K1) if and only if $\tilde{g}(\tilde{R}(A, B)(JC), JD) = \tilde{g}(\tilde{R}(A, B)C, D)$

for all $A, B, C, D \in \chi(\tilde{M})$

1. $\tilde{g}(\tilde{R}(X, Y)(J\partial_t), JW) = \tilde{g}(\tilde{R}(X, Y)\partial_t, W)$
 $\implies -t[g(R(X, Y)\xi, \varphi W) - \eta(Y)g(X, \varphi W) + \eta(X)g(Y, \varphi W)] = 0.$
 $\implies g(\varphi(R(X, Y)\xi - \eta(Y)X + \eta(X)Y), W) = 0$, for every W
 $\implies R(X, Y)\xi - \eta(Y)X + \eta(X)Y \in \ker \varphi.$

Thus, we have obtained

$$(12) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y \text{ modulo } \xi.$$

2. $\tilde{g}(\tilde{R}(X, Y)(JZ), J\partial_t) = \tilde{g}(\tilde{R}(X, Y)Z, \partial_t)$
 $\implies \tilde{g}(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y, -\frac{1}{t}\xi) =$
 $= \tilde{g}(R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, \partial_t)$
 $\implies -\frac{1}{t}t^2\eta(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y) = 0.$

Thus,

$$(13) \quad R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y \in \text{Ker } \eta$$

$$\begin{aligned}
3. \quad & \tilde{g}(\tilde{R}(X, Y)(JZ), JW) = \tilde{g}(\tilde{R}(X, Y)Z, W) \\
& \implies \tilde{g}(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y, \varphi W + t\eta(W)\partial_t) = \\
& \quad = \tilde{g}(R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, W).
\end{aligned}$$

After simplification by t^2 we obtain

$$\begin{aligned}
& g(R(X, Y)(\varphi Z), \varphi W) - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W) = \\
& \quad = g(R(X, Y)Z, W) - g(Y, Z)g(X, W) + g(X, Z)g(Y, W).
\end{aligned}$$

It follows

$$\begin{aligned}
(14) \quad & R(\varphi W, \varphi Z, X, Y) - R(W, Z, X, Y) = g(Y, \varphi Z)g(X, \varphi W) - \\
& \quad - g(X, \varphi Z)g(Y, \varphi W) + g(X, Z)g(Y, W) - g(Y, Z)g(X, W)
\end{aligned}$$

■

Remark 2.8. We immediately obtain (12) \longrightarrow (13) and (14) \longrightarrow (12).

Return to the formula (14). We interchange $(\varphi W, \varphi Z) \longleftrightarrow (X, Y)$, $(W, Z) \longleftrightarrow (X, Y)$ and then $Z \longleftrightarrow W$. One gets

$$\begin{aligned}
(*) \quad & \mathbf{R}(\mathbf{X}, \mathbf{Y}, \varphi \mathbf{Z}, \varphi \mathbf{W}) - \mathbf{R}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = \mathbf{g}(\mathbf{Y}, \varphi \mathbf{W})\mathbf{g}(\mathbf{X}, \varphi \mathbf{Z}) - \mathbf{g}(\mathbf{X}, \varphi \mathbf{W})\mathbf{g}(\mathbf{Y}, \varphi \mathbf{Z}) + \\
& \quad + \mathbf{g}(\mathbf{X}, \mathbf{W})\mathbf{g}(\mathbf{Y}, \mathbf{Z}) - \mathbf{g}(\mathbf{Y}, \mathbf{W})\mathbf{g}(\mathbf{X}, \mathbf{Z})
\end{aligned}$$

for all X, Y, Z, W in $\chi(M)$.

As consequences we have

$$\begin{aligned}
& R(\xi, Y, \xi, W) = g(Y, W) \\
& R(\xi, Y, Z, W) = R(\xi, Y, \varphi Z, \varphi W) = 0 \\
& R(X, Y, Z, W) - g(Y, W)g(X, Z) + g(X, W)g(Y, Z) = \\
& \quad = R(X, Y, \varphi Z, \varphi W) - g(Y, \varphi W)g(X, \varphi Z) + g(X, \varphi W)g(Y, \varphi Z),
\end{aligned}$$

where X, Y, Z and W are orthogonal to ξ .

Definition 2.9. We say that an almost contact metric manifold satisfies **(G1)**-identity if its curvature tensor verifies (*).

Proposition 2.10. The curvature tensor of a Sasakian manifold satisfies **(G1)** (see also Lemma 7.1 in [3]).

Proposition 2.11. Any contact manifold satisfying **(G1)** is Sasakian.

PROOF. It is known (e.g. Proposition 7.6 from [3]) that a contact manifold is Sasakian if and only if $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$, for all X and Y . ■

Return to the cone manifold \widetilde{M} . We give

Theorem 2.12. \widetilde{M} is (K2) if and only if

$$\begin{aligned}
(15) \quad & R(X, Y, Z, W) = R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W) \\
& \quad + g(X, Z)\eta(W)\eta(Y) - g(Z, Y)\eta(X)\eta(W)
\end{aligned}$$

PROOF. \widetilde{M} is (K2) if and only if

$$\tilde{R}(A, B, C, D) = \tilde{R}(JA, B, C, JD) + \tilde{R}(A, JB, C, JD) + \tilde{R}(A, B, JC, JD)$$

Three cases are essential:

- 1) $A = \partial_t$, $B = Y$, $C = \partial_t$, $D = W$ which is equivalent to $0 = 0$.
- 2) $A = \partial_t$, $B = Y$, $C = Z$, $D = W$

One has:

$$\begin{aligned}\tilde{R}(J\partial_t, Y, Z, JW) &= -\frac{1}{t}\tilde{R}(\xi, Y, Z, \varphi W) \\ \tilde{R}(\partial_t, JY, Z, JW) &= 0 \\ \tilde{R}(\partial_t, Y, JZ, JW) &= 0\end{aligned}$$

It follows that the right side is equal to:

$$-tg(\xi, R(Z, \varphi W)Y - g(\varphi W, Y)Z + g(Z, Y)\varphi W)$$

Since the left side vanishes, in this case we obtain

$$(16) \quad R(\xi, Y, Z, \varphi W) = \eta(Z)g(\varphi W, Y) \text{ for every } Y, Z, W \in \chi(M)$$

3) $A = X, B = Y, C = Z, D = W$. One has

$$\begin{aligned}\tilde{R}(JX, Y, Z, JW) &= \tilde{R}(\varphi X, Y, Z, \varphi W) \\ \tilde{R}(X, JY, Z, JW) &= \tilde{R}(X, \varphi Y, Z, \varphi W) \\ \tilde{R}(X, Y, JZ, JW) &= \tilde{R}(X, Y, \varphi Z, \varphi W)\end{aligned}$$

It follows that the right side is equal to

$$\begin{aligned}t^2[R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W)] + \\ + t^2[-g(\varphi W, \varphi Y)g(X, Z) + g(Z, Y)g(\varphi X, \varphi W)]\end{aligned}$$

while the left side equals to:

$$t^2R(X, Y, Z, W) + t^2[-g(W, Y)g(X, Z) + g(Z, Y)g(X, W)]$$

We get

$$(**) \quad \mathbf{R}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = \mathbf{R}(\varphi \mathbf{X}, \mathbf{Y}, \mathbf{Z}, \varphi \mathbf{W}) + \mathbf{R}(\mathbf{X}, \varphi \mathbf{Y}, \mathbf{Z}, \varphi \mathbf{W}) + \mathbf{R}(\mathbf{X}, \mathbf{Y}, \varphi \mathbf{Z}, \varphi \mathbf{W}) + \\ + \mathbf{g}(\mathbf{X}, \mathbf{Z})\eta(\mathbf{W})\eta(\mathbf{Y}) - \mathbf{g}(\mathbf{Z}, \mathbf{Y})\eta(\mathbf{X})\eta(\mathbf{W}).$$

It can be proved that previous relation implies 16. Hence the statement. ■

As consequences one has

$$\begin{aligned}R(\xi, Y, \xi, W) &= g(Y, W) \\ R(\xi, Y, Z, W) &= 0 \\ R(X, Y, Z, W) &= R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W)\end{aligned}$$

for all X, Y, Z, W orthogonal to ξ .

Definition 2.13. *We say that an almost contact metric manifold satisfies (G2)-identity if its curvature tensor verifies (**).*

Theorem 2.14. *The manifold \widetilde{M} is (K3) if and only if*

$$(17) \quad \begin{aligned}R(X, Y, Z, W) &= R(\varphi X, \varphi Y, \varphi Z, \varphi W) + g(X, Z)\eta(W)\eta(Y) - \\ &- g(Z, Y)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)\end{aligned}$$

for all $X, Y, Z, W \in \chi(M)$.

PROOF. \widetilde{M} is (K3) iff $\tilde{R}(A, B, C, D) = \tilde{R}(JA, JB, JC, JD)$ for all $A, B, C, D \in \chi(\widetilde{M})$. The essential cases are:

1) $A = \partial_t, B = Y, C = \partial_t, D = W$.

The left member vanishes and the right member is equal to $R(\xi, \varphi Y, \xi, \varphi W) - g(\varphi W, \varphi Y)$. We get

$$(18) \quad R(\xi, \varphi Y, \xi, \varphi W) = g(\varphi W, \varphi Y)$$

2) $A = \partial_t, B = Y, C = Z, D = W$.

The left member vanishes and the right member is equal to $R(\xi, \varphi Y, \varphi Z, \varphi W)$. We get

$$(19) \quad R(\xi, \varphi Y, \varphi Z, \varphi W) = 0$$

3) $A = X, B = Y, C = Z, D = W$.

The left member is equal to

$$t^2[R(X, Y, Z, W) - g(Z, X)g(W, Y) + g(Y, Z)g(X, W)]$$

and the right member is equal to

$$t^2[R(\varphi X, \varphi Y, \varphi Z, \varphi W) - g(\varphi W, \varphi Y)g(\varphi X, \varphi Z) + g(\varphi Y, \varphi Z)g(\varphi X, \varphi W)]$$

Hence (17) is proved. Remark that (17) implies both (18) and (19). ■

As consequences we have

$$R(\xi, Y, \xi, W) = g(Y, W)$$

$$R(\xi, Y, Z, W) = 0$$

$$R(X, Y, Z, W) = R(\varphi X, \varphi Y, \varphi Z, \varphi W)$$

for all $X, Y, Z, W \in \chi(M)$ orthogonal to ξ .

Definition 2.15. *We say that an almost contact metric manifold satisfies **G3**-identity if its curvature verifies relation (17).*

3. THE BOOTHBY WANG FIBRATION

Let M a $(2n + 1)$ -dimensional smooth manifold. A *contact form* on M is a 1-form η satisfying

$$\eta \wedge (d\eta)^n \neq 0.$$

We say that η endows on M a *contact structure*. It is clear that η induces an orientation on M and hence there exists a global non vanishing vector field ξ on M such that $\eta(\xi) = 1$. If ξ is *regular* in the sense of Palais (see [10]), then the contact structure (and also M) is called *regular*. If moreover M is compact, one can consider the space of all orbits of ξ , i.e. $N = M/\xi$ obtaining a smooth manifold. We have **Theorem A** ([5]). *Let (M, η) be a compact, regular, contact manifold. Then M is a principal circle bundle over N and η is a connection form of this bundle. The curvature form Θ of η defines a symplectic form on N . This fibration $S^1 \longrightarrow M \xrightarrow{\pi} N$ is called the Boothby-Wang fibration.*

Let Ω the symplectic 2-form of N , we denote by G the associated metric, i.e. $\Omega(X, Y) = G(X, JY)$ with J the almost complex structure.

In the following, we denote by X^\uparrow the lift of a vector field $X \in \chi(N)$. X^\uparrow is a horizontal vector field of M . On M a $(1, 1)$ tensor field φ can be defined, namely

$$(20) \quad \varphi X^\uparrow = (JX)^\uparrow, \quad \varphi \xi = 0.$$

We can easily see that

$$\varphi^2 = -I + \eta \otimes \xi$$

In this way, (φ, ξ, η) becomes an almost contact structure. The metric G can be lifted and hence one defines g on M as follows:

$$(21) \quad g = \pi^*G + \eta \otimes \eta$$

The metric g is compatible with the contact structure and $\xi = \eta^\#$.

Without loss of the generality one can suppose $d\eta = \pi^*\Omega$ and thus we have

$$g(X^\uparrow, \varphi Y^\uparrow) = G(X, JY) \circ \pi = \Omega(X, Y) \circ \pi = \pi^*\Omega(X^\uparrow, Y^\uparrow) = d\eta(X^\uparrow, Y^\uparrow)$$

In this way, (φ, ξ, η, g) becomes a contact metric structure on M .

If the symplectic structure of N derives from a Kaehlerian structure (J, G) , the obtained structure on M is Sasakian (i.e. contact and normal manifold). See e.g.[3]. But generally, a symplectic structure need not come from a Kaehlerian one. Yet, one can always find an almost Kaehlerian structure inducing it. In this case, the contact structure on the total space of a Boothby Wang fibration is K -contact, i.e. the vector field ξ is Killing, namely $\mathcal{L}_\xi g = 0$. It easily follows that the integral curves of ξ are geodesics.

It is easy to prove the relation

$$(22) \quad [X^\uparrow, Y^\uparrow] = [X, Y]^\uparrow - 2G(X, JY)\xi$$

for all $X, Y \in \chi(N)$.

Denote by $\overset{M}{\nabla}$ and $\overset{N}{\nabla}$ the Levi Civita connections on M and N , respectively. We immediately have:

$$g(\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow, Z^\uparrow) \circ \pi = G(\overset{N}{\nabla}_X Y, Z)$$

for any $X, Y, Z \in \chi(N)$. For the vertical part we shall compute $\eta(\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow)$:

$$\begin{aligned} 2g(\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow, \xi) &= X^\uparrow g(Y^\uparrow, \xi) + Y^\uparrow g(X^\uparrow, \xi) - \xi g(X^\uparrow, Y^\uparrow) + g([X^\uparrow, Y^\uparrow], \xi) + \\ &\quad + g([\xi, X^\uparrow], Y^\uparrow) + g(X^\uparrow, [\xi, Y^\uparrow]) = \\ &= \eta([X^\uparrow, Y^\uparrow]) - (\mathcal{L}_\xi g)(X^\uparrow, Y^\uparrow) \\ &= -2d\eta(X^\uparrow, Y^\uparrow). \end{aligned}$$

We obtain that

$$\eta(\overset{M}{\nabla}_{X^\uparrow} Y^\uparrow) \circ \pi = -G(X, JY).$$

In the following, we will ignore π , due to the isomorphism between the horizontal distribution of $T(M)$ and $T(N)$. Hence

$$(23) \quad \overset{M}{\nabla}_{X^\uparrow} Y^\uparrow = (\overset{N}{\nabla}_X Y)^\uparrow - G(X, JY)\xi.$$

In the same way, one can show

$$(24) \quad \overset{M}{\nabla}_{X^\uparrow} \xi = -\varphi X^\uparrow.$$

Denote by R^M and R^N the curvature tensors of M and N , respectively. Then

$$\begin{aligned} R^M(X^\uparrow, Y^\uparrow)Z^\uparrow &= (R^N(X, Y)Z)^\uparrow + g(Y^\uparrow, \varphi Z^\uparrow)\varphi X^\uparrow - g(X^\uparrow, \varphi Z^\uparrow)\varphi Y^\uparrow - \\ &\quad - 2g(X^\uparrow, \varphi Y^\uparrow)\varphi Z^\uparrow + \left\{ g(X^\uparrow, (\overset{M}{\nabla}_{Y^\uparrow} \varphi)Z^\uparrow) - g(Y^\uparrow, (\overset{M}{\nabla}_{X^\uparrow} \varphi)Z^\uparrow) \right\} \xi \end{aligned}$$

and hence

$$\begin{aligned} R^M(W^\uparrow, Z^\uparrow, X^\uparrow, Y^\uparrow) &= R^N(W, Z, X, Y) \circ \pi - 2g(X^\uparrow, \varphi Y^\uparrow)g(W^\uparrow, \varphi Z^\uparrow) + \\ &\quad + g(Y^\uparrow, \varphi Z^\uparrow)g(W^\uparrow, \varphi X^\uparrow) - g(X^\uparrow, \varphi Z^\uparrow)g(W^\uparrow, \varphi Y^\uparrow). \end{aligned}$$

Suppose that the base manifold N satisfies Gray identities. What are the corresponding curvature identities for the upstairs manifold M ?

If N is (K_1) then

$$\begin{aligned} R^M(X^\uparrow, Y^\uparrow, \varphi Z^\uparrow, \varphi W^\uparrow) - R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) &= \\ &= -g(Y^\uparrow, W^\uparrow)g(Z^\uparrow, X^\uparrow) - g(Y^\uparrow, \varphi W^\uparrow)g(Z^\uparrow, \varphi X^\uparrow) \\ &\quad + g(X^\uparrow, W^\uparrow)g(Z^\uparrow, Y^\uparrow) + g(X^\uparrow, \varphi W^\uparrow)g(Z^\uparrow, \varphi Y^\uparrow). \end{aligned}$$

If N is (K_2) then

$$\begin{aligned} R^M(\varphi X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) + R^M(X^\uparrow, \varphi Y^\uparrow, Z^\uparrow, W^\uparrow) + \\ + R^M(X^\uparrow, Y^\uparrow, \varphi Z^\uparrow, W^\uparrow) + R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, \varphi W^\uparrow) = 0. \end{aligned}$$

If N is (K_3) then

$$R^M(\varphi X^\uparrow, \varphi Y^\uparrow, \varphi Z^\uparrow, \varphi W^\uparrow) - R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) = 0.$$

These relations are exactly the defined Gray identities for almost contact metric manifolds for vector fields orthogonal to ξ .

4. PROPERTIES AND EXAMPLES

In their paper [9], D. Janssens and L. Vanhecke have studied curvature tensors for almost contact metric structures and defined *almost $C(\alpha)$ -manifolds*, namely those almost contact metric manifolds whose curvature tensor satisfies the following property:

$$\begin{aligned} \exists \alpha \in \mathbf{R} \text{ such that for all } X, Y, Z, W \in \chi(M) \\ R(X, Y, Z, W) = R(X, Y, \varphi Z, \varphi W) + \alpha \{ -g(X, Z)g(Y, W) + g(X, W)g(Y, Z) \\ + g(X, \varphi Z)g(Y, \varphi W) - g(X, \varphi W)g(Y, \varphi Z) \}. \end{aligned}$$

This means that manifolds satisfying the first Gray identity $(K_{1\varphi})$ in the sense of Bonome et al. are in fact $C(0)$ -manifolds, while that manifolds satisfying $(G1)$ are $C(1)$ -manifolds. Note that cosymplectic, Sasakian and Kenmotsu manifolds are respectively $C(0)$, $C(1)$ and $C(-1)$ manifolds (see Theorem 2.3, in [9]).

Let us come back to Gray identities for an almost Hermitian manifold.

It is known that $K1 \Rightarrow K2 \Rightarrow K3$ (see [7], §5). Consequently we have

Proposition 4.1. *For a class \mathcal{L} of almost contact metric manifolds, denote by \mathcal{L}_i the subclass of manifolds whose curvature satisfies G_i , $i = 1, 2, 3$. Then we have the following inclusions*

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \mathcal{L}.$$

As Gray remarked for Kaehlerian manifolds, we can say that *as i decreases, a manifold in \mathcal{L}_i resembles Sasakian manifold more closely.*

Proposition 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be a K -contact manifold satisfying $G1$ curvature identity. Then the manifold M is Sasakian.*

PROOF. By using Proposition 7.5 in [3], p.94, a K-contact manifold whose curvature satisfies $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ is Sasakian. But this last relation is a consequence of G1 identity. See also Proposition 2.11. ■

Proposition 4.3. *Let M be a contact metric manifold for which ξ belongs to the (κ, μ) -nullity distribution, namely its curvature satisfies*

$$(25) \quad R_{XY}\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

where $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ and κ, μ are constants. Suppose M satisfies (G1) identity. Then M is Sasakian.

PROOF. If M is (G1) then $R_{XY}\xi = \eta(Y)X - \eta(X)Y$ for all $X, Y \in \chi(M)$. Combining with the fact that ξ belongs to the (κ, μ) -nullity distribution we obtain

$$(\kappa - 1)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) = 0$$

for all $X, Y \in \chi(M)$. If $\mu \neq 0$ this implies $hY = \frac{1-\kappa}{\mu}Y$ for all $Y \in \ker \eta$. We know that h anticommutes with φ and hence one gets $\kappa = 1$. But using Theorem 7.7, p. 103 in [3] it follows that M is a Sasakian manifold. If $\mu = 0$ we immediately have $\kappa = 1$. ■

Proposition 4.4. *Let $(M, \varphi, \eta, \xi, g)$ be a contact metric manifold satisfying (G3) identity. Then M is K-contact.*

PROOF. Choose a φ -adapted local orthonormal frame on M , namely $\{X_i, \varphi X_i, \xi\}$, $i = 1, \dots, n$. Since M is (G3) the relation $R(X, \xi, Y, \xi) = g(X, Y)$ holds for all $X, Y \in \ker \eta$. Taking $X = Y = X_i$ (respectively $X = Y = \varphi X_i$) one immediately obtains $Ric(\xi, \xi) = 2n$, where Ric is the Ricci tensor on M . Now we use the fact that a contact metric manifold is K-contact if and only if the Ricci tensor in the direction of the characteristic vector field ξ is equal to $2n$ (Theorem, p.65, [2]). ■

4.1. An example of almost contact metric manifold satisfying G2 but not G1. The generalized Heisenberg group $H(p, 1)$ is defined as the set of matrices of real numbers having the form

$$a = \begin{bmatrix} 1 & A & c \\ 0 & I_p & {}^t B \\ 0 & 0 & 1 \end{bmatrix}$$

where I_p is the identity $p \times p$ matrix, $A = (a_1, \dots, a_p)$, $B = (b_1, \dots, b_p) \in \mathbf{R}^p$ and $c \in \mathbf{R}$. (Cf. [8].) $H(p, 1)$ is connected, simply connected nilpotent Lie group of dimension $2n + 1$. We will consider $p = 2$. A global system of coordinates (x^1, x^2, y^1, y^2, z) on $H(2, 1)$ is defined by $x^i(a) = a_i$, $y^i(a) = b_i$ for $i = 1, 2$ and $z(a) = c$. The global vector fields defined by

$$X_i = 2\frac{\partial}{\partial x^i}, \quad Y_i = 2\left(\frac{\partial}{\partial y^i} + x^i\frac{\partial}{\partial z}\right) \text{ for } i = 1, 2, \text{ and } \xi = 2\frac{\partial}{\partial z}$$

are left invariant. We take $\eta = \frac{1}{2}(dz - x^1 dy^1 - x^2 dy^2)$ and the metric

$$g = \frac{1}{4}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2) + \eta \otimes \eta.$$

By direct computations we obtain that $d\eta = -\frac{1}{2}(dx^1 \wedge dy^1 + dx^2 \wedge dy^2)$ and ξ is the characteristic vector field, namely $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. Moreover, the basis defined above is orthonormal: $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}$, $g(\xi, \xi) = 1$ and $g(X_i, Y_j) = g(X_i, \xi) = g(Y_i, \xi) = 0$. One has $[X_i, X_j] = 2\xi$ and the other brackets are equal to zero

and therefore it is easy to verify the Levi-Civita connection is given by the following formulas:

$$\begin{aligned}\nabla_{\xi} X_i &= -Y_i = \nabla_{X_i} \xi \\ \nabla_{\xi} Y_i &= X_i = \nabla_{Y_i} \xi \\ \nabla_{X_i} Y_i &= -\nabla_{Y_i} X_i = \xi\end{aligned}$$

for $i = 1, 2$, the other derivatives being zero. We compute also the Riemann-Christoffel curvature tensor field:

$$\begin{aligned}R(X_1, X_2, Y_1, Y_2) &= -1, & R(X_1, Y_2, X_2, Y_1) &= -1 \\ R(X_1, Y_1, X_2, Y_2) &= -2 & R(X_i, Y_i, X_i, Y_i) &= -3 \\ R(X_i, \xi, X_i, \xi) &= 1 & R(Y_i, \xi, Y_i, \xi) &= 1 \quad \text{for } i = 1, 2.\end{aligned}$$

The other values are zero or can be obtained from these ones. Define φ by:

$$\begin{aligned}\varphi X_1 &= \cos \theta Y_1 + \sin \theta Y_2 & \varphi X_2 &= \sin \theta Y_1 - \cos \theta Y_2 \\ \varphi Y_1 &= -\cos \theta X_1 - \sin \theta X_2 & \varphi Y_2 &= -\sin \theta X_1 + \cos \theta X_2\end{aligned}$$

and hence $(M, g, \varphi, \xi, \eta)$ is an almost contact metric manifold.

Proposition 4.5. *The structure is K-contact but not Sasakian.*

PROOF. For every $X, Y \in \chi(M)$ we have

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$$

On a Sasakian manifold, we should have $\nabla_X \xi = -\varphi X$ which implies in our case $\theta = 0$. So, in general, $H(2, 1)$ is not a Sasakian manifold. ■

Proposition 4.6. *On $H(2, 1)$ the G2 identity holds, while G1 identity doesn't.*

Straightforward computations to prove G2. Moreover, a K-contact manifold on which G1 holds is necessarily Sasakian. This is not the case. ■

4.2. Other examples. Let (N, \bar{g}, J) be an almost Hermitian manifold. Consider the warped product manifold $M = \mathbf{R} \times_f N$, where $f = f(\theta)$ is the warping function and θ is the global parameter on \mathbf{R} . Denote by $g = d\theta^2 + f^2(\theta)\bar{g}$ the Riemannian metric on M . Define also the global vector field $\xi = \frac{\partial}{\partial \theta}$, the 1-form $\eta = d\theta$ and the $(1, 1)$ tensor field φ by $\varphi X = JX$ if X is tangent to N and $\varphi \frac{\partial}{\partial \theta} = 0$. Thus (φ, ξ, η, g) is an almost contact metric structure on M . If $\bar{\nabla}$ and ∇ are the Levi Civita connections on N , respectively on M , we have

$$\nabla_{\xi} X = \nabla_X \xi = \frac{f'}{f} X, \quad \nabla_{\xi} \xi = 0, \quad \nabla_X Y = \bar{\nabla}_X Y - f f' \bar{g}(X, Y) \xi,$$

for all X, Y tangent to N .

The Riemann Christoffel curvature tensor is given by

$$\begin{aligned}(26) \quad R(W, \xi, X, Y) &= 0, & R(W, \xi, X, \xi) &= -\frac{f''}{f} g(X, W) \\ R(W, Z, X, Y) &= f^2 [\bar{R}(W, Z, X, Y) + \\ &\quad + (f')^2 (\bar{g}(X, Z) \bar{g}(Y, W) - \bar{g}(Y, Z) \bar{g}(X, W))] \end{aligned}$$

In order to have one of the three curvature identities we immediately have

$$\frac{f''}{f} = -1$$

which implies that $f = \alpha \cos \theta + \beta \sin \theta$ with α and β real constants. At this one can state the following

Proposition 4.7. *The manifold M is $G2$ (respectively $G3$) if and only if the almost Hermitian manifold N is $K2$ (respectively $K3$).*

PROOF. One has the following relations:

$$\begin{aligned} R(\varphi W, Z, X, \varphi Y) + R(W, \varphi Z, X, \varphi Y) + R(W, Z, \varphi X, \varphi Y) = \\ = f^2 [\bar{R}(JW, Z, X, JY) + \bar{R}(W, JZ, X, JY) + \bar{R}(W, Z, JX, JY)] \\ + (f')^2 f^2 (\bar{g}(X, Z)\bar{g}(Y, W) - \bar{g}(Y, Z)\bar{g}(X, W)) \end{aligned}$$

and

$$R(W, Z, X, Y) - R(\varphi W, \varphi Z, \varphi X, \varphi Y) = f^2 [\bar{R}(W, Z, X, Y) - \bar{R}(JW, JZ, JX, JY)].$$

Hence the statement. \blacksquare

Remark 4.8. *If $\dim N \geq 4$ then the manifold M cannot be $G1$.*

PROOF. Suppose M satisfies $G1$ identity. A straightforward computation gives

$$\begin{aligned} \bar{R}(W, Z, JX, JY) - \bar{R}(W, Z, X, Y) = (1 + (f')^2) [\bar{g}(JX, W)\bar{g}(JY, Z) - \\ - \bar{g}(JX, Z)\bar{g}(JY, W) + \bar{g}(Y, W)\bar{g}(X, Z) - \bar{g}(Y, Z)\bar{g}(X, W)]. \end{aligned}$$

Since f depends on θ (and it is not linear) while \bar{g} and \bar{R} do not, it follows that N is $K1$ and

$$\bar{g}(JX, W)\bar{g}(JY, Z) - \bar{g}(JX, Z)\bar{g}(JY, W) + \bar{g}(Y, W)\bar{g}(X, Z) - \bar{g}(Y, Z)\bar{g}(X, W) = 0$$

for all X, Y, Z, W tangent to N . This yields

$$(27) \quad \bar{g}(JY, Z)JX - \bar{g}(JX, Z)JY + \bar{g}(X, Z)Y - \bar{g}(Y, Z)X = 0.$$

If $\dim N \geq 4$ we can choose X and Y such that X , Y , JX and JY are linearly independent, so, the previous equality is impossible. \blacksquare

Example 4.9. On $M = \mathbf{R}^4 \times (-\pi/2, \pi/2)$ consider the global coordinates x, y, u, v and z . Consider the Riemannian metric $g = dz^2 + \cos^2 z (dx^2 + dy^2 + du^2 + dv^2)$ and the almost contact structure defined by: $\xi = \partial_z$, $\eta = dz$, $\varphi\partial_u = \partial_v$, $\varphi\partial_v = -\partial_u$ and $\varphi\partial_z = 0$. Then M is $G2$ but not $G1$.

Similarly for $M = \mathbf{R}^4 \times (0, \pi)$ and $g = dz^2 + \sin^2 z (dx^2 + dy^2 + du^2 + dv^2)$.

This kind of structure is called *sine-cone* and gives way to construct many geometric objects (e.g. nearly Kaehler structures starting from a 5-dimensional Sasaki Einstein manifold). Cf. [6].

Proposition 4.10. *Let N be a surface (which is automatically Kaehler) and consider the warped product manifold M as above. Then M satisfies $G1$.*

PROOF. The statement follows from the fact that a Kaehler manifold is $K1$ and the equation (27) is satisfied in dimension 2. \blacksquare

4.3. Hypersurfaces of almost Hermitian manifolds. Let $(\widetilde{M}, J, \widetilde{g})$ a $(2n+2)$ -dimensional Kaehler manifold, and let M be a totally umbilical (real) hypersurface in \widetilde{M} . Denoting by N the unit normal on M and let A be the Weingarten operator and h the scalar second fundamental form. As M is totally umbilical, we have that $AX = \beta X$, for all X tangent to M , with $\beta \in C^\infty(M)$.

It is well known the fact that on M we can define an almost contact metric structure (see e.g. [3]). More precisely, we take $\xi = -JN$ and for $X \in \chi(M)$ we decompose JX as:

$$JX = \varphi X + \eta(X)N.$$

Let g be the restriction of the metric \widetilde{g} on M . Denote by $\widetilde{\nabla}$ (respectively ∇) the Levi-Civita connection on \widetilde{M} (respectively on M). Then, by the formula of Gauss, one has

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)N$$

On the other hand we have $\widetilde{\nabla}_X \xi = -J\widetilde{\nabla}_X N = JAX = \varphi AX + \eta(AX)N$. Hence

$$\nabla_X \xi = \varphi AX \quad \text{and} \quad h(X, \xi) = \eta(AX).$$

Suppose now that M satisfies the (G3) identity. This implies that

$$(28) \quad R(X, \xi, Y, \xi) = g(X, Y) \quad \forall X, Y \in \text{Ker}(\eta)$$

We should compute $R(X, \xi)\xi = \nabla_X \nabla_\xi \xi - \nabla_\xi \nabla_X \xi - \nabla_{[X, \xi]}\xi$. Since M is totally umbilical, we have that $\nabla_X \xi = \beta \varphi X$. Thus $\nabla_\xi \xi = 0$. Then

$$\nabla_\xi \nabla_X \xi = \xi(\beta) \varphi X + \beta(\nabla_\xi \varphi)X + \beta \varphi \nabla_\xi X.$$

But

$$\nabla_\xi X = \beta \varphi X - [X, \xi]$$

and so

$$\nabla_\xi \nabla_X \xi = \xi(\beta) \varphi X + \beta(\nabla_\xi \varphi)X + \beta^2 \varphi^2 X - \beta \varphi [X, \xi].$$

It follows that

$$R(X, \xi)\xi = -\xi(\beta) \varphi X - \beta(\nabla_\xi \varphi)X + \beta^2 X.$$

Now, due the fact M is Kaehler, we have that

$$\widetilde{\nabla}(JY) = J\widetilde{\nabla}_X Y = J(\nabla_X Y + h(X, Y)N) = \varphi \nabla_X Y + \eta(\nabla_X Y)N - h(X, Y)\xi$$

On the other hand

$$\widetilde{\nabla}(JY) = \widetilde{\nabla}_X(\varphi Y + \eta(Y)N) = \nabla(\varphi Y) + h(X, \varphi Y)N + X\eta(Y)N - \eta(Y)\beta X.$$

Identifying the tangent and the normal parts of $\widetilde{\nabla}(JY)$ we obtain respectively

$$(29) \quad (\nabla_X \varphi)Y = \beta \eta(Y)X - \beta g(X, Y)\xi$$

$$(30) \quad (\nabla_X \eta)(Y) = -\beta g(X, \varphi Y).$$

Putting $X = \xi$ in (29) we have $(\nabla_\xi \varphi)Y = \beta \eta(Y)\xi - \beta g(\xi, Y)\xi = 0$ which implies

$$\nabla_\xi \varphi = 0.$$

Then

$$R(X, \xi)\xi = -\xi(\beta) \varphi X + \beta^2 X$$

From (28) we have that

$$g(\beta^2 X - X - \xi(\beta) \varphi X, Y) = 0, \quad \forall Y \in \text{ker}(\eta).$$

As X and φX are linearly independent (and belong to $\ker(\eta)$), we obtain that $\beta = \pm 1$. We obtain that

$$AX = \pm X, \quad \text{and} \quad h(X, Y) = \pm g(X, Y).$$

For $\beta = -1$ it follows that

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

By Theorem 6.14 in [3] this implies that M is Sasakian.

Proposition 4.11. *Let M a totally umbilical hypersurface of a Kaehler manifold \widetilde{M} endowed with the usual almost contact metric structure. If M satisfies the $G3$ identity, then M is a Sasakian manifold and hence M satisfies all G_i , for $i = 1, 2, 3$.*

More generally, if the second fundamental form of M is given by

$$h(X, Y) = \lambda\eta(X)\eta(Y) + \mu g(X, Y), \quad \forall X, Y \in \chi(M)$$

with λ and μ smooth functions on M , i.e. M is *totally quasi umbilical*, and if M satisfies $(G3)$ identity, then it is Sasakian. As consequence, there is no *cylindrical submanifold* satisfying $(G3)$ and whose second fundamental form being $h(X, Y) = \lambda\eta(X)\eta(Y)$.

Acknowledgements. The authors would like to thank Professor A. Bonome for discussions we had in Santiago de Compostela concerning this subject. They also wish to express their gratitude to Professor D.E. Blair for useful comments and suggestions during the preparation of this paper.

REFERENCES

- [1] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. A.M.S., 145(1969), 1-49.
- [2] D.E. Blair, *Contact Manifolds in Riemannian Geometry*, Springer Verlag, 1976.
- [3] D.E. Blair, *Riemannian geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, Birkhäuser Boston, 2002.
- [4] A. Bonome, L.M. Hervella and I. Rozas, *On the Classes of Almost Hermitian Structures on the Tangent Bundle of an Almost Contact Metric Manifold*, Acta Math. Hung., **56** (1990) (1-2), 29-37.
- [5] W.M. Boothby and H.C. Wang, *On Contact Manifolds*, Ann. of Math. **68** (1958), 721-734.
- [6] M. Fernández, S. Ivanov, V. Muñoz and L. Ugarte, *Nearly hypo Structures and compact Kaehler 6-manifolds with conical singularities*, arXiv:math.DG/0602160.
- [7] A. Gray, *Curvature Identities for Hermitian and Almost Hermitian Manifolds*, Tohoku Math. Journ., **28**(1976), 601-612.
- [8] J.C. Gonzales and D. Chinea, *Quasi-Sasakian Homogeneous Structures on the Generalized Heisenberg Group $H(p, 1)$* , Proc. A.M.S. 105 (1989) 1, 173-184.
- [9] D. Janssens and L. Vanhecke, *Almost Contact Structures and Curvature Tensors*, Kodai Math. J., **4** (1981), 1-27.
- [10] R.S. Palais, *A Global Formulation of the Lie Theory of Transformation Groups*, Mem. A.M.S. **22** (1957), 123p.

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